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Quasi-Interpolation on the 2-Sphere Using Radial Polynomials

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In this paper we consider a simple method of radial quasi-interpolation by polynomials on S^2 and present rates of convergence for this method on a wide range of smooth functions.

Key Words: sphere; radial functions; quasi-interpolation; convergence.

1. HARMONIC ANALYSIS AND SETS OF SMOOTH FUNCTIONS ON S^2

Let

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$$

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be the usual scalar product in Euclidean 3-space $\mathbb{R}^3$, and S^2 be the 2-dimensional unit sphere in $\mathbb{R}^3$, i.e.,

$$S^2 = \{x \mid x \in \mathbb{R}^3, \langle x, x \rangle = 1\}.$$

Let $d\mu$ be the normalized rotation invariant measure on the sphere, and

$$\|\varphi\|_p = \begin{cases} \left(\int_{S^2} |\varphi(x)|^p \, d\mu(x) \right)^{1/p}, & 1 \leqslant p < \infty, \\ \text{ess sup}\{ |\varphi(x)| \mid x \in S^2\}, & p = \infty. \end{cases}$$

Let $L_p = \{\varphi \mid \|\varphi\|_p < \infty\}$, and $U_p = \{\varphi \mid \|\varphi\|_p \leqslant 1\}$.

The space L_2 has the orthogonal decomposition

$$L_2 = \bigoplus_{k=0}^{\infty} H_k$$

where H_k is the space of *spherical harmonic polynomials* of degree k. It is known that H_k has dimension $2k+1$. Let $\{Y_{-k}^{(k)}, ..., Y_k^{(k)}\}$ be an orthonormal basis for H_k. For each $k \in \mathbb{N}$, H_k is an eigenspace of the Laplace–Beltrami operator for the sphere, Δ, corresponding to the eigenvalue $\gamma_k = -k(k+1)$, i.e., $\Delta Y_i^{(k)} = \gamma_k Y_i^{(k)}$, $i = -k, ..., k$.

A function Z_η is zonal with respect to a pole $\eta \in S^2$ if it is invariant under the action of all rotations σ of S^2 which fix η, i.e., $Z_\eta(x) = Z_\eta(\sigma x)$ for all $x \in S^2$, and $\sigma \in SO(3)$ with $\sigma\eta = \eta$. Then $Z_\eta(x) = \tilde{Z}(\langle x, \eta \rangle)$ for some $\tilde{Z}$ defined on $[-1, 1]$.

Equipped with a zonal kernel we may define the convolution

$$(\varphi * Z_\eta)(x) = \int_{S^2} \varphi(y) \, \tilde{Z}(\langle x, y \rangle) \, d\mu(y) = \int_{S^2} \varphi(y) \, Z_\eta(\sigma^{-1} y) \, d\mu(\eta),$$

where $x = \sigma\eta$.

The real zonal polynomial

$$Z_\eta^{(k)}(x) = \sum_{m=-k}^{k} \overline{Y_m^{(k)}(\eta)} \, Y_m^{(k)}(x) \tag{1}$$

is a kernel for orthogonal projection onto H^k. Furthermore,

$$\|Z_\eta^{(k)}\|_2^2 = \sum_{m=-k}^{k} |Y_m^{(k)}(\eta)|^2 = 2k+1, \qquad \forall \eta \in S^2.$$

The zonal harmonics have a simple expression in terms of the Legendre polynomials P_k, which can be defined in terms of the generating function

$$\frac{1}{(1-2\rho t+\rho^2)^{1/2}}=\sum_{k=0}^{\infty} \rho^k P_k(t),$$

where $0 \leqslant |\rho| < 1$ and $|t| \leqslant 1$. It is known that

$$Z_\eta^{(k)}(x)=\tilde{Z}^{(k)}(\langle x, \eta\rangle)=(2k+1)\, P_k(\cos\theta),$$

where $\cos\theta=\langle x, \eta\rangle$. In the sequel, where there is no possibility of confusion, we shall not make explicit reference to the pole η.

Each $\varphi \in L_1$ has a formal Fourier expansion in terms of complex spherical harmonics

$$\varphi \sim \sum_{k=0}^{\infty} \sum_{m=-k}^{k} c_{k,m}(\varphi)\, Y_m^{(k)}, \tag{2}$$

where the Fourier coefficients $c_{k,m}(\varphi)$ are given by

$$c_{k,m}(\varphi)=\int_{S^2} \varphi(z)\, \overline{Y_m^{(k)}(z)}\, d\mu(z).$$

A zonal function has a Fourier series in zonal polynomials

$$Z \sim \sum_{k=0}^{\infty} c_k(Z)\, Z^{(k)},$$

where

$$c_k(Z)=\frac{1}{2k+1} \int_{S^2} Z(z)\, Z^{(k)}(z)\, d\mu(z).$$

With the earlier definition of convolution we obtain the following familiar expression for the Fourier series of a convolution:

$$\varphi * Z \sim \sum_{k=0}^{\infty} c_k(Z) \sum_{m=-k}^{k} c_{k,m}(\varphi)\, Y_m^{(k)}. \tag{3}$$

We also have Young's inequality

$$\|\varphi * Z\|_q \leqslant \|\varphi\|_p\, \|Z\|_r, \tag{4}$$

where $1 \leqslant p, q, r \leqslant \infty$, $1/q=1/p+1/r-1$.

For more information on harmonic analysis on the sphere the reader should consult the Refs. [2, 3, 7–9, 11–13].

We shall introduce a wide range of smooth functions on a sphere in terms of multiplier operators, which, via (3), can often be realised as convolution operators. Given a sequence $\lambda = \{\lambda_k\}_{k \in \mathbb{N}}$, we shall say that the function f is in $\Lambda U_p \oplus \mathbb{R}$ if

$$f \sim c + \sum_{k=1}^{\infty} \lambda_k \sum_{m=1}^{2k+1} c_{k,m}(\varphi)\, Y_m^{(k)},$$

where $c \in \mathbb{R}$ and $\varphi \in U_p$. If the function $K \in L_1$ and

$$K \sim \sum_{k=1}^{\infty} \lambda_k Z^{(k)}$$

then the convolution $K * \varphi(x)$ is well defined and the function $f(x)$ can be written in the form

$$f(x) = K * \varphi(x) + c,$$

where $c \in \mathbb{R}$ is some constant. In this case we shall say $f \in K * U_p \oplus \mathbb{R}$.

We remark that the sets $\Lambda U_p \oplus \mathbb{R}$ are shift-invariant, i.e., for any $f \in \Lambda U_p \oplus \mathbb{R}$ and any $\sigma \in SO(3)$ we have

$$f_\sigma(x) = f(\sigma^{-1} x) \in \Lambda U_p \oplus \mathbb{R}.$$

2. ERROR ESTIMATES

In this section the letter C will denote a constant which will not necessarily have the same value at each occurrence.

As usual, we parametrize the points z on S^2 by their spherical coordinates $x = (\theta, \phi) \in [0, \pi] \times [0, 2\pi)$. In what follows we shall be concerned with approximation based on the equiangular grid points

$$\theta_s = \frac{\pi s}{2b}, \qquad \phi_t = \frac{\pi t}{b}, \qquad 0 \leqslant s \leqslant 2b-1, \quad 0 \leqslant t \leqslant 2b-1.$$

Let $x_{s,t} = (\theta_s, \phi_t)$.

For a continuous function f we consider the sequence of polynomials

$$T_{b-1}(f, x) = \sum_{k=0}^{b-1} \sum_{j=-k}^{k} \alpha_{k,j}(f)\, Y_j^{(k)}(x), \tag{5}$$

b even, where

$$\alpha_{k,j}(f) = \frac{\sqrt{2\pi}}{2b} \sum_{s=0}^{2b-1} a_s^{(b)} \sum_{t=0}^{2b-1} \overline{Y_j^{(k)}(\theta_s, \phi_t)}\, f(\theta_s, \phi_t),$$

and

$$a_s^{(b)} := \frac{2^{3/2}}{b} \sin\left(\frac{\pi s}{b}\right) \sum_{l=0}^{b/2-1} \frac{1}{2l+1} \sin\left((2l+1)\frac{\pi s}{b}\right). \tag{6}$$

This is the form of the discrete Fourier series suggested by Driscoll and Healy [4]. We remark here that in [4], b is a power of 2, while the discrete orthogonality results they prove are good for all even b. It can be shown (see [5]) that

$$0 \leqslant \sum_{l=0}^{b/2-1} \frac{1}{2l+1} \sin\left((2l+1)\frac{\pi s}{b}\right) \leqslant C, \tag{7}$$

where the constant C is independent of $b \in \mathbb{N}$ and $s = 0, ..., 2b-1$.

We may rewrite T_{b-1} as a quasi-interpolant

$$T_{b-1}(f, z) = \sum_{s=0}^{2b-1} \sum_{t=0}^{2b-1} f(\theta_s, \phi_t)\, L_{s,t}^{(b-1)}(z),$$

where

$$L_{s,t}^{(b-1)}(z) = \frac{\sqrt{2\pi}\, a_s^{(b)}}{2b} \sum_{k=0}^{b-1} \sum_{j=-k}^{k} \overline{Y_j^{(k)}(\theta_s, \phi_t)}\; Y_j^{(k)}(z)$$
$$= \frac{\sqrt{2\pi}\, a_s^{(b)}}{2b} \sum_{k=0}^{b-1} Z_{x_{s,t}}(z)$$

by (1). Thus, the quasi-Lagrange function for $x_{s,t}$, $L_{s,t}$, is zonal about $x_{s,t}$. Using Szegö [12, 4.5.3] we have the following very simple form for the quasi-Lagrange function,

$$L_{s,t}^{(b-1)}(z) = \left(\frac{\pi}{2}\right)^{1/2} a_s^{(b)}\, P_{b-1}^{(1,0)}(\langle z, x_{s,t}\rangle), \tag{8}$$

where $P_{b-1}^{(1,0)}$ is the Jacobi polynomial with weight $(1-x)$.

We shall say that the multiplier sequence $\lambda = \{\lambda_k\}_{k \in \mathbb{N}} \in A_p$ if

$$\lim_{k \to \infty} |\lambda_k| \, k^{1/2} = 0, \qquad p < 4/3,$$

$$\lim_{k \to \infty} |\lambda_k| \, k^{1/2} (\ln k)^{3/4} = 0, \qquad p = 4/3,$$

$$\lim_{k \to \infty} |\lambda_k| \, k^{2(1-1/p)} = 0, \qquad p > 4/3.$$

If

$$K \sim \sum_{k=1}^{\infty} \lambda_k Z^{(k)}$$

for $\lambda \in A_p$ we shall (with a slight abuse of notation) say $K \in A_p$.

The main result of this article is

THEOREM 1. *Let K be an integrable radial* (*zonal*) *function on S^2 with Fourier series*

$$K \sim \sum_{k=1}^{\infty} \lambda_k Z^{(k)},$$

where $\{\lambda_k\}_{k \in N} \in A_p$. Then,

$$\sup_{f \in K * U_p \oplus \mathbb{R}} \| f - T_{b-1}(f, \cdot) \|_\infty \leqslant C b^{1/2} \sum_{k=b}^{\infty} |\Delta^2 \lambda_k| \, k^{1+2/p},$$

$$b \to \infty, \quad 1 \leqslant p \leqslant \infty.$$

Remark 1. By a proper choice of the multiplier sequence $\{\lambda_k\}_{k \in \mathbb{N}}$ we obtain the following examples. For examples (b) and (c) the results detailed in the Appendix are required.

(1) If

$$\lambda_k = (k(k+1))^{-\alpha/2}, \qquad \alpha > 0, \quad k \in \mathbb{N},$$

then the resulting convolution class $K * U_p \oplus \mathbb{R}$ is the Sobolev's class W_p^α; see [2, 6]. In this case we have

$$\sup_{f \in W_p^\alpha} \| f(z) - T_b(f, z) \|_\infty \leqslant C b^{((-\alpha+2)/p)+1/2}.$$

(2) The function $f(x)$, $x \in \mathbb{R}^3$ is analytic at $x_0 \in R^3$ if there is a series

$$\sum_{\alpha} a_\alpha (x - x_0)^\alpha$$

which converges to $f(x)$ in some closed ball $|x - x_0| \leqslant \rho$, $\rho > 0$. It is known (see [10, p. 494]) that if $f(x)$ is analytic on the sphere S^2 then $|Y_k(f, x)| \leqslant Ke^{-vk}$, where $Y_k(f, x) = Z^{(k)} * f(x)$, $k \in \mathbb{N}$ is the spherical harmonic of f of degree k, $K > 0$ and $v > 0$ are some fixed constants. So, if we put

$$\lambda_k = e^{-\alpha k}, \qquad \alpha > 0, \quad k \in \mathbb{N},$$

then the resulting convolution class $K * U_p \oplus \mathbb{R}$ will be a set of analytic functions on S^2. In this situation we have

$$\sup_{f \in K * U_p \oplus \mathbb{R}} \|f(z) - T_b(f, z)\|_\infty \leqslant Ce^{-\alpha b} b^{3/2 + 2/p}.$$

(3) Let

$$\lambda_k = e^{-\alpha k^r}, \qquad \alpha > 0, \quad r > 0, \quad k \in \mathbb{N}.$$

Then the set $K * U_p \oplus \mathbb{R}$ is a set of entire (if $r > 1$) or infinitely differentiable (if $0 < r < 1$) functions on S^2 and

$$\sup_{f \in K * U_p \oplus \mathbb{R}} \|f(z) - T_b(f, z)\|_\infty \leqslant Ce^{-\alpha b^r} b^{\min\{1, r\} + 1/2 + 2/p}.$$

In order to prove the main result of this paper we require several lemmas.

LEMMA 1. *For any zonal (radial) polynomial P_b of degree $\leqslant b$ on S^2 we have*

$$\sum_{0 \leqslant s, t \leqslant 2b-1} \sin\theta_s \, |P_b(\theta_s, \phi_t)| < \frac{2b^2(1+\pi)^2}{\pi^2} \int_{S^2} d\mu(z) \, |P_b(z)|.$$

Proof. Let P_b be a polynomial on S^2, of degree $\leqslant b$. P_b can be thought of as a trigonometric polynomial $P_b(z) = P_b(\theta, \phi)$ of the same degree. Let us fix $\phi = \phi_t$ and consider the integral

$$I(P_b, \phi_t) := \int_0^\pi |P_b(\theta, \phi_t)| \sin\theta \, d\theta = \sum_{s=0}^{2b-1} \int_{\theta_s}^{\theta_{s+1}} |P_b(\theta, \phi_t)| \sin\theta \, d\theta.$$

Since $|P_b(\theta, \phi_t)| \sin \theta$ is continuous function then, due to the mean value theorem, there is a $\tilde{\theta}_s \in [\theta_s, \theta_{s+1}]$ such that

$$I(P_b, \phi_t) = \frac{\pi}{2b} \sum_{s=0}^{2b-1} |P_b(\tilde{\theta}_s, \phi_t)| \sin \tilde{\theta}_s.$$

Now, due to the Minkowsky inequality

$$\begin{aligned}
&\sum_{s=0}^{2b-1} |P_b(\theta_s, \phi_t)| \sin \theta_s - \frac{2b}{\pi} \int_0^{\pi} |P_b(\theta, \phi_t)| \sin \theta \, d\theta \\
&\quad = \sum_{s=0}^{2b-1} |P_b(\theta_s, \phi_t)| \sin \theta_s - \sum_{s=0}^{2b-1} |P_b(\tilde{\theta}_s, \phi_t)| \sin \tilde{\theta}_s \\
&\quad \leqslant \sum_{s=0}^{2b-1} |P_b(\theta_s, \phi_t) \sin \theta_s - P_b(\tilde{\theta}_s, \phi_t) \sin \tilde{\theta}_s| \\
&\quad \leqslant \sum_{s=0}^{2b-1} \int_{\theta_s}^{\theta_{s+1}} |(P_b(\theta, \phi_t) \sin \theta)'| \, d\theta \\
&\quad = \int_0^{\pi} |(P_b(\theta, \phi_t) \sin \theta)'| \, d\theta \\
&\quad < \int_{-\pi}^{\pi} |(P_b(\theta, \phi_t) \sin \theta)'| \, d\theta. \qquad (9)
\end{aligned}$$

Applying the Bernstein inequality to (9) we have

$$\begin{aligned}
&\sum_{s=0}^{2b-1} |P_b(\theta_s, \phi_t)| \sin \theta_s - \frac{2b}{\pi} \int_0^{\pi} |P_b(\theta, \phi_t)| \sin \theta \, d\theta \\
&\quad < (b+1) \int_{-\pi}^{\pi} |P_b(\theta, \phi_t) \sin \theta| \, d\theta \\
&\quad = (b+1) \left(\int_0^{\pi} |(P_b(\theta, \phi_t) \sin \theta| \, d\theta + \int_{-\pi}^{0} |(P_b(\theta, \phi_t) \sin \theta| \, d\theta \right). \qquad (10)
\end{aligned}$$

Suppose $P_b(z)$ has pole $z' = (\theta', \phi')$. Then $P_b^{z'}(z) = \tilde{T}_b(\cos \gamma)$, where $\tilde{T}_b$ is a polynomial of degree $\leqslant b$ on $[-1, 1]$ and $\cos \gamma = \langle z, z' \rangle = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')$ (see [7]). Thus

$$\int_{-\pi}^{0} |P_b^{(\theta', \phi')}(\theta, \phi_t) \sin \theta| \, d\theta$$

$$= \int_0^{\pi} |\tilde{T}_b(\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - (\pi + \phi'))) \sin \theta| \, d\theta$$

$$= \int_0^{\pi} |P_b^{(\theta', \pi + \phi')}(\theta, \phi_t) \sin \theta| \, d\theta. \tag{11}$$

Combining (10) and (11) we see that

$$\left(1 + \frac{\pi(b+1)}{2b}\right)^{-1} \frac{\pi}{2b} \sum_{s=0}^{2b-1} |P_b^{(\theta, \phi)}(\theta_s, \phi_t)| \sin \theta_s$$

$$< \left(\int_0^{\pi} |P_b^{(\theta', \phi')}(\theta, \phi_t) \sin \theta| \, d\theta + \int_0^{\pi} |P_b^{(\theta', \pi + \phi')}(\theta, \phi_t) \sin \theta| \, d\theta \right). \tag{12}$$

Let us fix now θ_s, $0 \leqslant s \leqslant 2b - 1$. Using the same arguments as above we can show that

$$\frac{\pi}{b} \sum_{t=0}^{2b-1} |P_b(\theta_s, \phi_t)| \leqslant (1 + \pi) \int_0^{2\pi} |P_b(\theta_s, \phi)| \, d\phi. \tag{13}$$

Since the norm on S^2 is shift invariant then from (12) and (13) we have

$$\frac{\pi^2}{2b^2} \sum_{0 \leqslant s, \, t \leqslant 2b-1} \sin \theta_s \, |P_b(\theta_s, \phi_t)|$$

$$< (1 + \pi) \left(1 + \frac{\pi(b+1)}{2b}\right) \int_{S^2} d\mu(z) \, |P_b(z)|$$

$$\leqslant (1 + \pi)^2 \int_{S^2} d\mu(z) \, |P_b(z)|. \quad \blacksquare$$

Let us define the Lebesgue function L_{b-1} for T_{b-1} by

$$L_{b-1}(\theta, \phi) := \sum_{0 \leqslant s, \, t \leqslant 2b-1} |L_{s, t}^{(b-1)}(\theta, \phi)|.$$

The next statement gives us order of growth of Lebesgue constants.

Lemma 2. *The Lebesgue constants of quasi-interpolation satisfy*

$$\max_{z \in S^2} L_{b-1}(z) \leqslant C b^{1/2}, \qquad b \to \infty.$$

Proof. From (8) we have

$$L_{b-1}(z)=\left(\frac{\pi}{2}\right)^{1/2} \sum_{1\leqslant s,\, t\leqslant 2b-1} a_s^{(b)}\, |P_{b-1}^{(1,0)}(\langle z, x_{s,t}\rangle)|,$$

using (6) and (7). For all $0\leqslant s, t\leqslant 2b-1$ we have, again using (6) and (7), $b^{-1}a_s^{(b)}\leqslant Cb^{-2}\sin\theta_s$, $b\to\infty$. Therefore

$$L_{b-1}(z)\leqslant Cb \sum_{1\leqslant s,\, t\leqslant 2b-1} b^{-2}\sin\theta_s\, |P_{b-1}^{(1,0)}(\langle z, x_{s,t}\rangle)|.$$

Now, from Lemma 1 and the last inequality we see that

$$\max_{z\in S^2} L_{b-1}(z)\leqslant Cb\, \|P_{b-1}^{(1,0)}(\langle\cdot, \eta\rangle)\|_1\leqslant Cb^{1/2},$$

for any fixed $\eta\in S^2$, since $\|P_{b-1}^{(1,0)}(\langle\cdot,\eta\rangle)\|_1\asymp b^{-1/2}$ (see [6, 7]).

LEMMA 3. *Let $K\in A_p$, with $p\geqslant 1$. Then there is a radial (zonal) polynomial $t_{b-1}(K,\cdot)$ such that*

$$\|K(\cdot)-t_{b-1}(K,\cdot)\|_p\leqslant C\sum_{k=b}^{\infty}|\Delta^2\lambda_k|\, k^{3-2/p}, \qquad b\to\infty.$$

Proof. We will need some facts about Cesàro means. For the generating function $(1-\rho)^{-m-1}$ we have

$$\frac{1}{(1-\rho)^{m+1}}=\sum_{s=0}^{\infty} A_s^m\,\rho^n=\sum_{s=0}^{\infty}\frac{\Gamma(s+m+1)}{\Gamma(m+1)\,\Gamma(s+1)}\,\rho^s,$$

$$A_s^m\leqslant Cs^m, \quad s\to\infty.$$

Let

$$C_n^\delta=\frac{1}{A_n^\delta}\sum_{k=0}^{n} A_{n-k}^\delta Z^{(k)}$$

be the Cesàro kernel of order n and index δ. It is known [7] that

$$\|C_n^\delta\|_p\leqslant C\begin{cases} n^{1/2-\delta}, & p<4/(3+2\delta),\\ n^{1/2-\delta}(\ln n)^{3/4}, & p=4/(3+2\delta),\\ n^{2(1-1/p)}, & p>4/(3+2\delta).\end{cases} \tag{14}$$

Applying the Abel transform we have the following representation which is valid, in L_p, for any $K \in A_p$:

$$K = \sum_{k=1}^{\infty} \Delta^2 \lambda_k k C_k^1,$$

where $\Delta^2 \lambda_k := \lambda_k - 2\lambda_{k+1} + \lambda_{k+2}$.

As in [3, 6], let us consider the sequence of polynomials

$$t_b = \sum_{k=1}^{b} \Delta^2 \lambda_k k C_k^1.$$

Using the asymptotic estimates for Cesàro kernels (14) we can complete the proof, since

$$\left\| K - \sum_{k=1}^{b-1} \Delta^2 \lambda_k k C_k^1 \right\|_p \leqslant C \sum_{k=b}^{\infty} |\Delta^2 \lambda_k| k^{3-2/p}. \quad ▮$$

COROLLARY 1. *Let $K \in A_r$ where $r = 1/(1-(1/p-1/q)_+)$. Then there is a sequence of polynomials $t_b(f)$ such that*

$$\sup_{f \in \Lambda U_p \oplus \mathbb{R}} \|f - t_{b-1}(f)\|_q \leqslant C \sum_{k=b}^{\infty} |\Delta^2 \lambda_k| \, k^{3-2/r}, \qquad b \to \infty.$$

Corollary 1 follows from the Young inequality (4) and Lemma 3.

Proof of Theorem 1. Observing that the operator $T_{b-1}(f, \cdot)$ is a linear projector onto the space of polynomials of degree $\leqslant b-1$ (see [4]) we can apply the Lebesgue inequality

$$|f(z) - T_{b-1}(f, z)| \leqslant E_{b-1}(f)(1 + L_{b-1}(z)), \qquad z \in S^2, \tag{15}$$

where $f \in C(S^2)$, $E_{b-1}(f) = \inf\{\|f - t_{b-1}\|_\infty \mid t_{b-1} \in \mathscr{T}_{b-1}\}$ and $\mathscr{T}_{b-1}$ is the subspace of polynomials of degree $\leqslant b-1$. The theorem follows via Corollary 1, Lemma 1, and (15). ▮

3. EXAMPLES

In this section we give a number of examples which suggests that the convergence rate of Theorem 1 is pessimistic in terms of power scale for Sobolev classes. We approximate the zonal function $f_\beta(\theta) = (\cos\theta)^\beta$ for different non integer values of β. By [122.2.10, 22.13.8, and 22.13.9], and Parseval's identity, $f_\beta \in W_2^\alpha$ for $\beta > \alpha - 1/2$. Theorem 1 predicts a rate of convergence $b^{-\beta+1+\varepsilon}$ for any $\varepsilon > 0$.

We compute an approximation to the error

$$e_b(f) = \sup_{z \in \Lambda} |f(z) - T_b(f)(z)|,$$

where $z = (\theta_z, \phi_z) \in \Lambda$ if and only if $\theta_z = \pi j/20$ for some j and $\phi_z = l\pi/10$ for some l. In other words, we compute the pointwise error on a 20 by 20 grid on the sphere. Then, we obtain an approximate rate of convergence $r_b = \log_2(e_{2b}/e_b)$.

In Figs. 1–3 we show the errors and associated approximate rate of convergence, all given to three significant figures, for different values of β.

b	e_b	r_b
4	0.187	-0.261
8	0.156	-0.367
16	0.121	-0.433
32	8.96×10^{-2}	-0.465
64	6.49×10^{-2}	-0.485
128	4.64×10^{-2}	

FIG. 1. $\beta = 0.5$.

b	e_b	r_b
4	5.28×10^{-2}	-1.04
8	2.57×10^{-2}	-1.09
16	1.21×10^{-2}	-1.15
32	5.42×10^{-3}	-1.20
64	2.36×10^{-3}	-1.24
128	1.00×10^{-3}	

FIG. 2. $\beta = 1.25$.

b	e_b	r_b
4	2.91×10^{-2}	-1.31
8	1.17×10^{-2}	-1.33
16	4.66×10^{-3}	-1.40
32	1.77×10^{-3}	-1.45
64	6.48×10^{-4}	-1.47
128	2.34×10^{-4}	

FIG. 3. $\beta = 1.5$.

APPENDIX

Let α, r, $\beta > 0$. Then, for sufficiently large b,

$$\sum_{k=b}^{\infty} k^{\beta} e^{-\alpha k^r} \leqslant C b^{\beta+1-\min\{1,r\}} e^{-\alpha b^r}.$$

Proof. First suppose that $r \geqslant 1$. Then, since $(k+1)^r \geqslant k^r + 1$, we have $e^{-\alpha(k+1)^r} \leqslant e^{-\alpha} e^{-\alpha k^r}$. Applying this inequality m times we see that $e^{-\alpha(k+m)^r} \leqslant e^{-m\alpha} e^{-\alpha k^r}$. Therefore,

$$\begin{aligned}\sum_{k=b}^{\infty} k^{\beta} e^{-\alpha k^r} &\leqslant e^{-\alpha b^r} \sum_{k=0}^{\infty} (b+k)^{\beta} e^{-\alpha k} \\ &= b^{\beta} e^{-\alpha b^r} \sum_{k=0}^{\infty} (1+k/b)^{\beta} e^{-\alpha k} \\ &\leqslant C b^{\beta} e^{-\alpha b^r},\end{aligned}$$

since the final sum converges for all positive α, β.

Now suppose $0 < r < 1$. Then, for sufficiently large b,

$$\sum_{k=b}^{\infty} k^{\beta} e^{-\alpha k^r} \leqslant b^{\beta} e^{-\alpha b^r} + \int_b^{\infty} t^{\beta} e^{-\alpha t^r}\, dt. \tag{16}$$

Integrating by parts we have

$$\begin{aligned}\int_b^{\infty} t^{\beta} e^{-\alpha t^r}\, dt &= \int_b^{\infty} t^{\beta+1-r} t^{r-1} e^{-\alpha t^r}\, dt \\ &= -\frac{1}{\alpha r} [t^{\beta+1-r} e^{-\alpha t^r}]_b^{\infty} + \frac{\beta+1-r}{\alpha r} \int_b^{\infty} t^{\beta-r} e^{-\alpha t^r}\, dt \\ &\leqslant \frac{b^{\beta+1-r} e^{-\alpha b^r}}{\alpha r} + \frac{\beta+1-r}{\alpha r} b^{-r} \int_b^{\infty} t^{\beta} e^{-\alpha t^r}\, dt \\ &\leqslant \frac{b^{\beta+1-r} e^{-\alpha b^r}}{\alpha r} + \frac{1}{2} \int_b^{\infty} t^{\beta} e^{-\alpha t^r}\, dt,\end{aligned}$$

for large enough b. In other words, as $b \to \infty$,

$$\int_b^{\infty} t^{\beta} e^{-\alpha t^r}\, dt \leqslant 2 \frac{b^{\beta+1-r} e^{-\alpha b^r}}{\alpha r},$$

and the result follows for $0 < r < 1$ by (16). ∎

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